QUANTITATIVE OPERATORS IN MATHEMATICAL MODELING

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In this report, we present cases where students constructed new quantities through operating on quantities that does not fit the definitions of existing theories on quantitative operations. As a result, we identified five quantitative operators—operators that can be used on single qualities in order to transform the quantity to a new quantity—students used while constructing mathematical models for real-world scenarios.

Keywords: Quantitative Reasoning, Mathematical Modeling, Operations on Quantities

Mathematical modeling is an important skill for students to learn. However, it is still a challenging subject for students (Stillman et al., 2010; Jankvist & Niss, 2020). In an effort to mitigate some of these challenges, recently, mathematical modeling scholars have adopted theories from quantitative reasoning (Thompson, 1994; 2011) to operationalize mathematical modeling competencies (e.g. Czoher et al., 2022; Larsen, 2013; Roan & Czocher, 2022). In this adaptation, quantities are viewed as building blocks of a mathematical model (Larsen, 2013). That is, a new quantity can be constructed through operating on one or more existing quantities. As a result, a mathematical model maybe viewed as a network of such operations on quantities (Thompson, 1990). If this perspective on *mathematical models* and *mathematical modeling* is to be taken to investigate students' mathematical modeling, then more work needs to be done on the mechanisms involved in the conception of a new quantity through operating on existing quantity or quantities. These mechanisms have been explicated by scholars as quantitative operations (Thompson, 1990) and (co)variational reasoning (Carlson et al., 2002) through investigating, predominantly, K-12 students' mathematical reasonings. However, it is still not clear exactly how theories from quantitative reasoning explain students' reasoning as students mathematically model dynamic, complex situations, especially those that require differential equations. For example, students have constructed rate of change through operating on existing quantities in ways that cannot be explained by the current theories of quantitative reasoning (Kularajan & Czocher, 2022). In this report, we share examples from a study of students' reasoning during mathematical modeling that demonstrate the need for amending and extending theories of quantitative reasoning to include *quantitative operators*.

Operation on Quantities

Quantities are conceptual entities that exist in the mind of an individual. Thompson (1994) defined *quantity* as a mental construct consisting of three interdependent entities: an object, a measurable attribute, and a *quantification*. Quantification involves conceiving a measurable attribute of an object and a unit of measure and forming a proportional relationship between the attribute's measure and the unit of measure (Thompson, 2011). While objects are constructions taken as given, the attributes that one conceives as measurable are imbued by the individual conceiving them (Thompson, 1994). Thompson (1990) explains this phenomenon through the ideas of motion and distance moved in an amount of time. For example, for a young child watching a cat running to hunt a bird in the backyard, the cat probably is an object, and the young child may have imbued the attribute motion to the running cat. However, for this young

child, the running cat probably does not have the attribute distance moved in a corresponding amount of time.

A relationship among measurable attributes is established through operating on quantities. Thompson (1994) defines *quantitative operation* as the “mental operation by which one conceives a new quantity in relation to one or more already-conceived quantities” (p.10). As a result of a quantitative operation a *quantitative relationship* is created: the quantities operated upon along with the quantitative operation are in relation to the result of operating (Thompson, 1994). In other words, a quantitative relationship is the “conception of three quantities, two of which determine the third by a quantitative operation (Thompson, 1990, p. 12).” Examples of quantitative operations include *combining two quantities additively*, *comparing two quantities additively, combining two quantities multiplicatively*, *comparing two quantities multiplicatively*, *instantiate a rate*, *generalize a ratio*, and *composing two rates or ratios* (Thompson, 1994). For example, how many more cats visited my backyard on Saturday than on Sunday is a quantity that we may construct by *additively comparing* the number of cats that visited my backyard on Saturday and the number of cats that visited my backyard on Sunday. At the same time, we may construct the total number of cat-bird interactions on Saturday during the time period 9am to 5pm by *instantiating a rate* of 10 cat-bird interactions per hour for 8 hours.

Although Thompson (1994) defined quantitative operations as the mental operations on “one or more already-conceived quantities” to construct a new quantity, the definition of a quantitative relationship (Thompson, 1990) and the examples given for quantitative operations (Thompson, 1994) emphasize operations on two quantities to conceive a third new quantity. Therefore, it is not clear through the definition of quantitative operations (Thompson 1994a), quantitative relationships (Thompson, 1990), and the examples of quantitative operations, whether mental operations performed on one quantity to construct a new quantity fall within the scope of Thompson’s quantitative operations. In addition, students may engage in operations on quantities without clear evidence of the operations having a situationally relevant quantitative meaning, but the resultant quantity has a quantitative meaning for the student. We refer to these borderline instances as pseudo-quantitative operations and present examples in our findings.

Methods

Data for this report were drawn from a larger study of effective scaffolding for promoting modeling competencies. We worked with 34 undergraduate STEM majors who were enrolled in or had already taken differential equations at the time of the interviews. The students participated in 10 hour-long task-based clinical interviews (Goldin, 2000) where they developed mathematical models for real-world systems. We present examples, to illustrate our case, from four students’ work—Ivory, Szeth, Pattern, and Winnow—on The Cats and Birds Task, The Tropical Fish Task, The Pruning Task, and The Tuberculosis Task.

> *The Cats and Birds Task:* Cats, our most popular pet, are becoming our most embattled. A national debate has simmered since a 2013 study by the Smithsonian’s Migratory Bird Center and the U.S. Fish and Wildlife Service concluded that cats kill up to 3.7 billion birds and 20.7 billion small mammals annually in the United States. The study blamed feral “unowned” cats but noted that their domestic peers “still cause substantial wildlife mortality.” In this problem, we will build a model (step-by-step) that predicts the species’ population dynamics, considering the interaction of the two species.

The Tropical Fish Task: To regulate the pH balance in a 300L tropical fish tank, a buffering agent is dissolved in water and the solution is pumped into the tank. The strength of the buffering solution varies according to $1 - e^{\frac{-t}{20}}$ grams per liter. The buffering solution enters the tank at a rate of 5 liters per minute. Create an expression that models how quickly the amount of buffering agent in the tank is changing at any moment in time.

The Pruning Task: Imagine you have a hedge in your garden of some size, S, and you want it to increase its size even more. You hire a gardener for some advice on growing this particular plant. She advises you that the overall rate of growth will depend both on the extent of pruning and on the regrowth rate, which is particular to the plant species and environmental conditions. Both rates can be measured as a percentage of the size of the plant. The pruning rate can be adjusted to result in a target overall growth rate. Can you derive a model for the rate of change of the size of the plant?

The Tuberculosis Task: Tuberculosis (TB) is a serious infectious disease caused by a bacterium that originated in cattle, but can affect all mammals including humans. It typically affects the lungs, causing a general state of illness, coughing, and eventual death. Many infected individuals carry a latent (inactive) infection for a long time before their lungs succumb to the damage caused by the bacteria. The disease is highly contagious; it is spread from person to person when an infected individual coughs, spits, speaks, or sneezes. Because transmission rates are so high, TB outbreaks are frequently associated with poverty conditions – locations where overcrowding is common. In these communities, spread (rate of new infections) can be very high and decimate a community rapidly. Imagine a community where sick and well members move about freely among one another. Create a mathematical model for the rate that the disease will spread through the community.

The interviews were retrospectively analyzed to construct second-order accounts (Steffe & Thompson, 2000) of students' reasonings via inferences made from students' observable activities such as verbal descriptions, language, written work, discourse, and gestures. The retrospective analysis consisted of multiple passes of the data to arrive at examples that illustrate the different ways students engaged in Pseudo-Thompsonian Quantitative Operations. First, we watched the videos in MAXQDA in chronological order and paraphrased each interview by chunks. Next, we created accounts of students' mathematics and the reasons they attributed to their mathematics. Next, we reviewed the accounts and the videos at the same time and refined our accounts by adding details using theories from quantitative reasoning. We credited a student to have instantiated a quantity if we were able to infer from his reasonings that he had conceived an object, attribute, and a measurement process for the attribute. As evidence of student to have conceived a measurement process, we checked if at least one of the quantifications criteria was met (see Czocher & Hardison, 2021). We used segments of transcripts, where the students engaged in quantitative reasoning along with inscriptions and gestures as evidence for our claims. Next, from these accounts, we selected instances where the students constructed a new quantity by operating on a singular quantity or the operation itself (to construct the new quantity) did not have clear evidence of a situationally relevant quantitative meaning. Finally, we refined our second-order accounts by triangulating with utterance and gestures to support our claims.

Findings

We identified five examples where students engaged in pseudo-quantitative operations: (1)

constructing rate of change through taking the derivative, (2) constructing total amount through taking the integral, (3) constructing percent through considering parts of a whole, (4) constructing amount through considering a proportion of the whole, and (5) constructing rate of change through negation. We illustrated the first example in Kularajan & Czocher (2022), where we showed two modelers taking the derivative to construct the rate of change of the bird population due to predation by cats, in The Cats and Birds Task. In this report, we present examples of (2)-(5).

Taking the Integral to Construct Total Amount

To illustrate this operation, we present Ivory's work from The Tropical Fish Task. In the Tropical Fish Task, Ivory was working towards constructing a model for the amount of buffering agent in the tank at time t. To accomplish this goal, Ivory first constructed expression 1 to represent the rate at which the amount of buffering agent enters the tank.

$$m_E(t) = 5 \cdot (1 - e^{\frac{-t}{20}}) \tag{1}$$

In expression 1, Ivory defined $m_E(t)$ as the rate at which the amount of buffering agent enters the tank at time t. After constructing expression 1, she stated that she would take the integral of $m_E(t)$ in order to construct an expression for the amount of buffering agent in the tank at time t. Following this reasoning, Ivory constructed the expression below to represent the amount of buffering agent inside the tank at time t.

$$M(t) = \int m_E(t) \tag{2}$$

The interviewer pointed out that the expression 2, as written, only accounts for the amount of buffering agent that had entered the tank and ignores the amount of buffering agent that exits the tank. In response, Ivory modified expression 2 to the one shown in Figure 1 while confidently voicing that expression 2 "does work." In Figure 1, Ivory defined $m_L(t)$ as the rate at which the amount of buffering agent leaves the tank at time t. Through the expression in Figure 1, we infer that Ivory constructed the amount of buffering agent in the tank at time t, by *additively comparing* the amount of buffering agent that enters the tank and the amount of buffering agent that leaves the tank.

$$M(t) = \int M_E(t) - \int M_L(t)$$

Figure 1: Ivory's model for the amount of buffering agent in the tank at time t

We infer that Ivory first constructed the rates at which the amount of buffering agent enters and leaves the tank to operate on them further through taking the integral to construct the amount of buffering agent that enters the tank and the amount of buffering agent that leaves the tank, respectively. Even though Ivory constructed a quantity that had situationally relevant meaning to her (amount of buffering agent that enters (leaves) the tank at time t), the operation (taking the integral) on the singular quantity (rate at which the amount of buffering agent enters (leaves) the tank) did not have clear evidence of a situationally relevant quantitative meaning. For Ivory, taking the integral was an operation that could be performed on the measurable attribute rate of change to construct the amount.

Envisioning Parts of the Whole to Construct Percent

To illustrate this operation, we present Szeth's work from the Pruning task. Szeth first constructed Expression 3 where Szeth defined R' as the rate at which the plant would be growing, P as the "pruning," G' as the "regrowth rate," and E as "environmental conditions."

$$R' = P + G' + E \tag{3}$$

After Szeth constructed expression 3, he mathematized R' and G' as $R' = \frac{S}{100}$ and $G' = \frac{S}{100}$ because "both rates can be measured as a percentage of the size of the plant." We interpret that when Szeth read the task "the overall rate of growth [of the plant] will depend both on the extent of pruning and on the regrowth rate…Both rates can be measured as a percentage of the size of the plant," he interpreted "both rates" to be the rate at which the plant is growing (R') and the regrowth rate (G'), as oppose to regrowth rate and the extent of pruning. Therefore, Szeth constructed R' and G' as a percentage of the size of the plant, S, through considering $\frac{1}{100}^{th}$ of the size of the whole plant S. In this instance, for Szeth, considering $\frac{1}{100}^{th}$ of the size of the plant S was an operation on the quantity S in order to construct the new quantity "regrowth rate." Szeth said that he would substitute $R' = \frac{S}{100}$ and $G' = \frac{S}{100}$ in expression 3. Following this, the conversation below was exchanged among us.

Interviewer: You have $R' = \frac{S}{100}$ and $G' = \frac{S}{100}$. So, does that say that R' and G' are both equal to each other, or can they be different percentages?

Szeth: I guess it does say they're equal. I wouldn't take them to be equal. In real life perspective, they are supposed to be different things.

Interviewer: And how would you modify it so that they're not equal?

Szeth: Think I would just get rid of this [*scratches off* $R' = \frac{S}{100}$], because G' is already in an equation that affects R'. So, if I put this, let's substitute that [*pointing at* $G' = \frac{S}{100}$] into there [*pointing at* G' *in* $R' = P + G' + E$], then it's still true that this rate [*referring to* R'] can be measured as a percentage of the size. It still be involved in this equation up here.

In the above excerpt, when the interviewer asked Szeth if R' and G' are equal, Szeth responded that "in real life…they are supposed to be different things." By that, we interpret that he meant R' and G' measure different qualities of the plant, that may (or may not) have different values. We take this as evidence that for Szeth considering a fraction of the size of the plant—in particular considering $\frac{1}{100}^{th}$ of S—was a mental operation performed on the size of the plant, to construct the quantities that can be written as a percentage as a size of the plant. In response, when we asked how he would modify R' and G' such that they wouldn't be equal, Szeth indicated that he would substitute $G' = \frac{S}{100}$ in expression 3 and modified expression 3 as below.

$$S' = P + \frac{S}{100} + E \tag{4}$$

In expression 4, Szeth replaced $R' = S'$ and indicated that S' is implicitly written as a percentage of the size of the plant because S' is written in terms of $G' = \frac{S}{100}$.

In this example, we illustrated how Szeth constructed the quantity G'—"regrowth rate"—through operating on the quantity S—"the size of the plant"—by considering a portion of S. In other

words, we illustrated how Szeth constructed G' by structurally conceiving percent through imagining a part, in particular $\frac{1}{100}^{th}$, of a whole size of the plant.

Envisioning a Proportion of the Whole to Construct Amount

To illustrate this operation, we present Pattern's work from the Cats and Birds task. In The Cats and Birds Task, Pattern was working towards constructing a model for the number of cat-bird encounters that result in a bird's death. To accomplish this goal, Pattern first constructed an expression to calculate the maximum number of cat-bird encounters at time t (Figure 2(a)). Pattern was then asked to consider how he might modify his expression to account for the fact that only a proportion of the maximum encounters are realized. In response to this, Pattern elected to multiply the number of maximum number of cat-bird encounters at time t by some percentage α, shown in Figure 2(b). Pattern explained his reasoning for multiplying by α as follows.

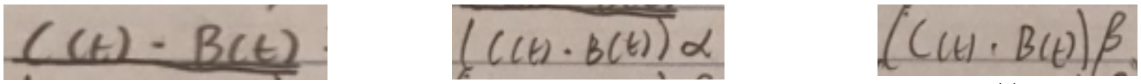
> Pattern: So this [*referring to the expression in Figure 2(a)*] is the total possible encounters that could possibly happen if perfect conditions are met for each cat to meet each bird, and then you're going to take a percentage of that total, and that would be your total.

In this instance, Pattern constructed the number of encounters that actually happened by envisioning a proportion of the maximum number of cat-bird encounters at time t. Here, for Pattern, α acted as a multiplicative scaling factor to quantify the proportion of cat-bird interaction that were realized. There was no clear evidence that α had a situationally relevant quantitative meaning for Pattern. That is, we were not able infer a situational referent (an object) and the attribute α was measuring.

Later, Pattern was asked to adapt his model (Figure 2(a)) to account for the fact that sometimes a bird might escape. In response to the request, Pattern decided to multiply the maximum number of cat-bird encounters at time t by some percentage β as shown in Figure 1(c). Pattern explained his decision to multiply by β as follows.

> Pattern: Because what I did here was I made it really easy for myself by creating this baseline [*referring to $C(t) \cdot B(t)$)*]. And from here, you can... Since they're not giving it to me, I can add whatever I want. So that gives me the freedom of being like, well, since you want to know how many birds die, we can just create this percentage [*referring to* $\big(C(t) \cdot B(t)\big) \cdot \beta$].

In this instance, Pattern constructed the number of encounters that result in a bird's death by taking a different proportion of the maximum number of cat-bird encounters at time t. Through Pattern's reasoning above, we interpret that, $C(t) \cdot B(t)$ acted as a "baseline" to consider different proportions of the maximum number of cat-bird interactions—number of cat-bird interactions that realized and number of cat-bird interactions that resulted in a dead bird. Pattern accomplished this by using multiplicative scaling factors (e.g. α% and β%, respectively) that reduced the size of the whole—maximum number of cat-bird interactions. In both of these instances, Pattern constructed new quantities by considering a proportion of the "baseline" via using multiplicative scaling factors . However, α% and β% were not associated with a discernable or situationally relevant attribute.



(a) (b) (c)

Figure 2: Pattern's models for (a) maximum number of cat-bird encounters, (b) number of actual cat-bird encounters, (c) number of cat-bird encounters that resulted in a birds death

Negating a Rate of Change to Construct a New Rate of Change

To illustrate this operation, we present Winnow's work from The Tuberculosis Task. Winnow constructed expression 5 as a model for the rate of change of the sick people with respect to time.

$$\frac{dS}{dt} = \frac{m}{S(t) \times H(t)} \times H(t) \tag{5}$$

In expression 5, Winnow defined $H(t)$ as the number of healthy people at time t, $S(t)$ as the number of sick people at time t, and m as the number of contacts between healthy and sick people that actually occur. When the interviewer asked for a model for the rate of change of the healthy people with respect to time. Winnow noted immediately that the rate of change of the healthy people "would be a negative number because the healthy people would be—the number of healthy people would be decreasing." Following this reasoning, Winnow constructed an initial model for the rate of change of healthy people with respect to time as shown in expression 6.

$$\frac{dH}{dt} = -\frac{m}{S(t) \times H(t)} \times S(t) \tag{6}$$

After constructing expression 6, Winnow validated his model by checking against specific conditions, $S(t) = 10$ and $H(t) = 10$. He was happy that substituting for $S(t) = 10$ and $H(t) = 10$ in expressions 5 and 6 yielded the same value, but the negation of one another, asserting that the rate of change with respect to time for sick people would be the same value as the rate of change with respect to time for healthy people.

To perturb Winnow, the Interviewer asked him to validate his model against the values $S(t) = 2$ and $H(t) = 10$. Realizing that $\frac{dS}{dt}$ and $\frac{dH}{dt}$ would not yield the equal values, Winnow modified his model for the rate of change of healthy people with respect to time to be $\frac{dH}{dt} = -\frac{dS}{dt}$. Winnow justified his modification to his model as "the number of new sick people and the decrease in healthy people should be the same." Through this we infer that winnow negated the rate of change of sick people with respect to time to construct the rate of change of healthy people with respect to time, because for Winnow, the number of *new* sick people, directly corresponds to the *decrease* in the number of healthy people. However, there was no clear evidence whether the " $-$ " in $\frac{dH}{dt} = -\frac{dS}{dt}$ represented a quantity, let alone -1, for Winnow.

Discussion

We introduced the construct pseudo-quantitative operations to account for five instances of modelers constructing quantitative relationships without clear evidence that the operations performed on quantities had situationally relevant quantitative meaning. In this report, we presented four such instances. First, we illustrated how Ivory constructed the amount of buffering agent entering and leaving the tank by taking the integral of the rate at which the buffering agent enters the tank and the rate at which the buffering agent leaves the tank, respectively. Next, we presented an example of how Szeth constructed the "regrowth rate" of the plant by considering a fraction of the whole size of the plant, by using a multiplicative scaling

factor of $\frac{1}{100}$. Third, we illustrated how Pattern constructed the number of birds that died due to predation by cats by *shrinking* the total number of cat-bird interactions that realized through using a multiplicative scaling factor α. Finally, we presented how Winnow considered the negation of the rate of change of the sick people with respect to time in order to construct the rate of change of healthy people with respect to time.

What do we mean by *the operations on quantities having no situationally relevant quantitative meaning?* Recall the example where Winnow constructed the rate of change of healthy people with respect to time by negating the rate of change of sick people with respect to time. If Winnow had indicated that the $-$ in $\frac{dH}{dt} = -\frac{dS}{dt}$ represented $\frac{-1\, healthy\ person}{1\, sick\ person}$ — a measurable attribute of the system of healthy and sick people—then we would have credited Winnow to have engaged in the *multiplicative combination* of two quantities (number of people removed from the healthy population for each person getting sick and the rate of change of the sick people with respect to time), making it a quantitative operation as Thompson (1990) defined it. For Winnow, the $-$ in $\frac{dH}{dt} = -\frac{dS}{dt}$ acted as an operator to transform $\frac{dS}{dt}$, in turn quantifying $\frac{dH}{dt}$.

Similarly, when Pattern constructed the number of birds that died due to predation by cats by shrinking the total number of cat-bird interactions that realized through using a multiplicative scaling factor α, we were not able to infer whether for Pattern α carried any situationally relevant quantitative meaning. We would have credited Pattern to have engaged in a quantitative operation—*instantiating a rate*—between α and $C(t) \cdot B(t)$, if he had shown clear evidence that α measured the number of cat-bird interactions that resulted in a dead bird for every 100 cat-bird interactions, an attribute of the system of cats and birds. However, for Pattern α acted as an operator that *shrank* the size the total number of possible cat-bird interaction to represent a subset of that amount. Likewise, when Szeth constructed a percentage of the size of the plant as $\frac{S}{100}$, the $\frac{1}{100}$ acted as an operator to transform S in order quantify the "regrowth rate." At the same time, for Ivory, the integral acted as an operator to transform a *rate* to an *amount*; there was no clear evidence whether the integral involved the coordination of two quantities.

Even though taking the derivative, integral, using a multiplicative scaling factor, and negating in and of itself may not be credited as quantitative operations, they are operations that the students performed on existing quantities to construct a new quantity. In addition, the aforementioned operations on existing quantities are an image that is prevalent in undergraduate learners and the result of that action is often useful for mathematical modeling. Adhering to the definitions of quantitative operations, as Thompson defined it, instances such as illustrated in this report will be missed. Therefore, to investigate students' quantitative reasoning in mathematical modeling, we propose to extend the definition of *operations on quantities* to include operations on singular quantities to construct new quantities through the aid of *quantitative operators*. We identified five such *quantitative operators*: $\frac{d\,\blacksquare}{dt}$, $\int \blacksquare$, $\alpha \times \blacksquare$, $\frac{1}{100} \times \blacksquare$, and $-\blacksquare$. These *quantitative operators* can be viewed as functions that take in a quantity and output a new quantity, where a quantity itself can be operationalized as a function consisting of three variables (object, attribute, and quantification). The inclusion of these *quantitative operators* allows us to make better sense of the mechanisms involved in the construction of new quantities in students' mathematical modeling. Our findings reaffirm that quantitative reasoning is essential to mathematical modeling, but also caution that not every parameter, variable, or operation in the

student's work needs to signify a situational referent in order for their modeling activities to be productive.

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